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PARTITIONS OF PLANAR SETS INTO SMALL TRIANGLES

Andrew M. ODLYZKO

AT & T Bell Laboratories, Murray Hill, NJ 07974

János PINTZ

Mathematical Institute, Hungarian Academy of Sciences, H-1053 Budapest, Hungary

Kenneth B. STOLARSKY

Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.

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Given $3n$ points in the unit square, $n \geqslant 2$, they determine n triangles whose vertices exhaust the given $3n$ points in many ways. Choose the n triangles so that the sum of their areas is minimal, and let $a^*(n)$ be the maximum value of this minimum over all configurations of $3n$ points. Then $n^{-\frac{1}{2}} \ll a^*(n) \ll n^{-\frac{1}{9}}$ is deduced using results on the Heilbronn triangle problem. If the triangles are required to be *area disjoint* it is not even clear that the sum of their areas tends to zero; this open question is examined in a slightly more general setting.

1. Introduction

Given a set $P = \{p_1, \ldots, p_{3n}\}$ of $3n$ points in a convex planar body Σ, let $a(n, P, \Sigma)$ denote the smallest sum of areas of a collection $\Delta = \{\delta_1, \ldots, \delta_n\}$ of n triangles such that

(i) the vertices of the δ_i exhaust the $3n$ points of P, and

(ii) the intersection of any δ_i with any δ_j, $j \neq i$, has zero area.

A system of triangles satisfying (i) and (ii) shall henceforth be called a *disjoint triangle partition.*

Let $a(n, \Sigma)$ denote the supremum of $a(n, P, \Sigma)$ over all sets $P \subseteq \Sigma$ of $3n$ points, and let $a(n)$ denote the supremum of $a(n, \Sigma)/A(\Sigma)$ over all convex planar sets Σ of positive area. We establish for $n \geqslant 2$ that

$$n^{-\frac{1}{2}} \ll a(n) \leqslant 0.8 + O(n^{-1}) \leqslant 0.9. \tag{1.1}$$

Elementary considerations (see Section 5) show that the limit of $a(n)$ as $n \to \infty$ exists, but the authors do not know whether or not it is zero, although they can improve on (1.1) to some extent.

If the disjointness requirement (ii) is dropped, so that we ask for that collection of triangles $\Delta = \{\delta_1, \ldots, \delta_n\}$ whose vertices exhaust the points of P and which has minimal sum of areas of the triangles, the problem becomes somewhat more

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tractable. Denote the function in this case by $a^*(n)$, so that $a(n) \geqslant a^*(n)$. We show that for every $\epsilon > 0$,

$$n^{-\frac{1}{2}} \ll a^*(n) \ll n^{-\frac{1}{8}+\epsilon}. \tag{1.2}$$

As is explained in Section 3, it seems likely that the lower bound in (1.2) is close to the true value of $a^*(n)$.

Our problem is related to Heilbronn's triangle problem, which asks for an estimate of the area of the smallest triangle determined by any three out of n points located in the unit square. As a result of the deep work of Schmidt [11], Roth [6–10], and Komlós, Pintz and Szemerédi [5] it is known [5] that there is always a triangle of area

$$\ll n^{-\frac{8}{7}+\epsilon}, \tag{1.3}$$

for every $\epsilon > 0$. On the other side, Erdös' early example [6] of a configuration without a triangle of area $\ll n^{-2}$ was recently improved by Komlós, Pintz and Szemerédi [4], who showed that there are configurations without triangles of area $\ll n^{-2} \log n$. Our proofs rely on a modification of the Erdös construction and on the Komlós, Pintz and Szemerédi result (1.3).

2. The upper bound

We first establish that

$$a(2, P, \Lambda) \leqslant 0.8 A(\Lambda) \tag{2.1}$$

where Λ is the convex hull of the given set P of $3n = 6$ points. The argument splits into cases according to whether

$$b = |P \cap \partial\Lambda|, \tag{2.2}$$

the number of points of P on the boundary $\partial\Lambda$ of the convex hull Λ, is six, five, or at most four.

Whenever there are two collections of triangles $\{\delta_1, \ldots, \delta_n\}$ and $\{\gamma_1, \ldots, \gamma_n\}$ satisfying (i) and (ii) such that each δ_i is area-disjoint from each γ_j, clearly one of the collections has total area at most

$$\tfrac{1}{2}A(\Lambda), \tag{2.3}$$

where Λ is the convex hull of P. For $3n = 6$ and $b \leqslant 4$, it is easy to verify that two such collections exist.

For $b = 6$ label the vertices in clockwise order as

$$\{p_1, \ldots, p_6\} = \{B, C, D, E, F, G\}.$$

Then the three disjoint triangle partitions

$$\{BCD, EFG\}, \{CDE, FGB\}, \{DEF, GBC\} \tag{2.4}$$

have total combined area at most $2A(\Lambda)$, so some one of them has total area at most

$$\tfrac{2}{3}A(\Lambda). \tag{2.5}$$

Finally, suppose that $b=5$ and

$$\{p_1, \ldots, p_5\} = \{B, C, D, E, F\} \tag{2.6}$$

are the vertices, in clockwise order, of a convex pentagon, inside of which is $p_6 = G$. Then the hull Λ splits into five triangles

$$GBC, GCD, GDE, GEF, GFB, \tag{2.7}$$

so without loss of generality

$$A(GDE) \geq \tfrac{1}{5}A(\Lambda). \tag{2.8}$$

Now G lies in one of the triangles

$$BFE, BED, BDC; \tag{2.9}$$

without loss of generality it lies in BFE or BED. Thus

$$\{BCD, GEF\} \tag{2.10}$$

forms a disjoint triangulation of area at most

$$A(\Lambda) - A(GDE) \leq 0.8A(\Lambda) \tag{2.11}$$

by (2.8). This proves (2.1).

We now establish the right inequality of (1.1). Let m be a positive integer. To find a disjoint triangulation for $P \subseteq \Sigma$ where $|P| = 6m+3$, start with a vertical line t outside and to the left of Σ. We may assume (rotate Σ if necessary) that none of the $\leq\binom{3n}{2}$ lines determined by the $3n$ points is parallel to t. Then find $M = 2m+2$ lines $t_1, \ldots, t_M$ parallel to t, each to the right of the previous one, such that

(a) in each closed strip T_i formed by t_i, t_{i+1} there are 3 distinct elements of P, namely p_{i1}, p_{i2}, p_{i3}, and

(b) the union of all $\{p_{i1}, p_{i2}, p_{i3}\}$ is P.

Remarks. (1) Some of the t_i may coincide due to multiple points.

(2) If Σ is a rectangle with two sides parallel to t, the triangles p_{i1}, p_{i2}, p_{i3} immediately provide a disjoint triangulation of area at most $0.5A(\Lambda)$.

The procedure now is to combine pairs of adjacent strips to produce roughly half the original number of parallel strips, with each new strip containing 6 distinct points of P. Of course, there will be one strip T^* left over that contains only 3 points. Since there are $m+1$ ways of doing this, we can insure that T^* intersects Σ in a set of area at most

$$A(\Sigma)/(m+1) \tag{2.12}$$

and hence the triangle formed by the points in T^* has at most this much area.

The first result of this section shows that the six points in each of the remaining m strips T_i can be triangulated so that at most 0.8 of each $T_i \cap \Sigma$ is covered. Let α be the fraction of the area of Σ that lies in T^*. Then the fraction of Σ covered by the resulting disjoint triangulation is at most

$$0.8(1-\alpha)+\alpha \leqslant 0.8+\frac{0.2}{m+1} \leqslant 0.9. \tag{2.13}$$

If $|P|=6m$ the argument simplifies, and we have the better upper bound 0.8. This proves the right inequality of (1.1).

3. No disjointness requirement

We now use the Komlós, Pintz and Szemerédi result (1.3) to prove the upper bound of (1.2). We first prove a general result which gives an upper bound for $a^*(n)$ in terms of any upper bound for Heilbronn's triangle problem. Since the asymptotic behavior of $a^*(n, \Sigma)/A(\Sigma)$ is the same for all convex Σ, we will take Σ to be the unit square.

Notation. Let $\Delta(n)$ denote the maximum possible value of the minimum of the areas of the triangles $p_i p_j p_k$ (taken over all selections of three out of n points $p_1, \ldots, p_n$), where the maximum is taken over all distributions of $p_1, \ldots, p_n$ in the unit square. Let $\tilde{\Delta}(n)=\Delta(3n)$.

Theorem. *If* $1 \leqslant f(n) \leqslant n^{\lambda}$ *is monotonically increasing and*

$$\tilde{\Delta}(n) \leqslant f(n)/n^{1+\lambda}, \tag{3.1}$$

where $0<\lambda \leqslant 1$, *then*

$$a^*(n) \leqslant \frac{500}{\lambda}(f(n)n^{-\lambda})^{1/(1+\lambda)}. \tag{3.2}$$

Proof. The upper bound $a^*(n) \leqslant 1$ implies the claimed assertion for $n \leqslant 24$, for example. We suppose that the assertion is true for all $j<n$, where $n \geqslant 25$. We divide the square into 4 smaller squares of area $\frac{1}{4}$ and choose one of these little squares, say Q', which contains at least $3n/4$ points. By possibly shrinking Q' to a square Q'' we can assume that the number of points in Q'' is $3k$ with $(n-3)/4 \leqslant [n/4] \leqslant k \leqslant n-1$ (if we additionally make the convention that points on the boundary of Q'' are to be considered (separately) to belong to Q'' or not according to our decision). Now we proceed as follows.

(1) We choose the smallest triangle in $Q-Q''$, afterwards the smallest remaining triangle in $Q-Q''$, etc., until the number of remaining points is $3[M]$, where

$$M=\frac{(f(n) \cdot n)^{1/(1+\lambda)}}{100}.$$

(Note $M \leqslant n/100$.) The sum of the areas of these triangles is

$$\leqslant \sum_{l=[M]+1}^{n-k} \tilde{\Delta}(l) \leqslant f(n) \sum_{l=[M]+1}^{\infty} \frac{1}{l^{1+\lambda}} \leqslant \frac{2f(n)}{\lambda M^{\lambda}} \leqslant \frac{200}{\lambda} (f(n)n^{-\lambda})^{1/(1+\lambda)}. \tag{3.3}$$

(2) To each of the $3[M]$ remaining points in $Q - Q''$ we associate successively 2 points from Q'' so that the resulting $3[M]$ triangles (with disjoint vertices) all have areas

$$\leqslant \frac{1}{2} (\sqrt{2})^2 \sin \frac{2\pi}{\frac{3n}{4} - 6M} \leqslant \frac{10}{n}.$$

The sum of the areas of these triangles is therefore

$$\leqslant \frac{30M}{n} < (f(n)n^{-\lambda})^{1/(1+\lambda)}. \tag{3.4}$$

(3) Finally for the remaining $3k - 6[M]$ points in Q'', where

$$\frac{n}{5} \leqslant \frac{n-3}{4} - \frac{n}{50} \leqslant k - 2[M] \leqslant n - 1,$$

we have by our inductive hypothesis $k - 2[M]$ triangles with total area

$$\leqslant \frac{1}{4} a^*(k - 2[M]) \leqslant \frac{1}{4} \cdot \frac{500}{\lambda} \left(f(n) \left(\frac{n}{5} \right)^{-\lambda} \right)^{1/(1+\lambda)}$$

$$\leqslant \frac{125}{\lambda} \cdot \sqrt{5} (f(n)n^{-\lambda})^{1/(1+\lambda)}. \tag{3.5}$$

Summing the areas of all these triangles ((3.3)–(3.5)) we obtain

$$a^*(n) \leqslant \frac{500}{\lambda} (f(n)n^{-\lambda})^{1/(1+\lambda)}. \quad \square$$

Remark. Since we know by [5] that $\Delta(n) \ll n^{-\frac{9}{8}}$, for example, we are entitled to suppose $\lambda \geqslant \frac{1}{8}$ and so the constant $500/\lambda$ can be replaced by 4 000.

Using the inequality

$$\Delta(n) \ll e^{c\sqrt{\log n}} \cdot n^{-\frac{8}{7}}$$

proved in [5], we obtain

Corollary 1. $a^*(n) \ll e^{c\sqrt{\log n}} \cdot n^{-\frac{1}{8}}$.

Although Heilbronn's conjecture was disproved in [4] by showing that

$$\Delta(n) \gg n^{-2} \log n,$$

one may conjecture that $\Delta(n) \ll n^{-2+\epsilon}$, however.

Corollary 2. *The conjecture $\Delta(n) \ll n^{-2+\epsilon}$ implies that $a^*(n) \ll n^{-\frac{1}{2}+\epsilon}$.*

The strongest possible conjecture $\Delta(n) \ll n^{-2} \log n$ would imply $a^*(n) \ll n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}$.

These results show that probably the inequality $a^*(n) \gg n^{-\frac{1}{2}}$ cannot be improved sigmificantly. Our theorem also shows that a proof of a relation of type $\overline{\lim}_{n\to\infty} a^*(n)n^{\frac{1}{2}} = \infty$ would imply $\overline{\lim}_{n\to\infty} \Delta(n)n^2 = \infty$, so it would lead to a new disproof of Heilbronn's conjecture (if the inequality $\Delta(n) \gg n^{-2} \log n$ is not used in course of the proof, naturally). This connection shows that the following problem might be interesting.

Problem. Is it true that $a^*(n) \ll n^{-\frac{1}{2}}$?

4. The lower bound

Since

$$a^*(n) \leqslant a(n), \tag{4.1}$$

the left side of (1.1) follows immediately from the left side of (1.2), which we shall establish after a preliminary lemma.

Lemma. *Let p be an odd prime, and let $z_1, \ldots, z_p$ be the lattice points in*

$$[0, p-1] \times [0, p-1]$$

whose coordinates are congruent modulo p to those of

$$(k, k^2), \quad 0 \leqslant k \leqslant p-1.$$

Then (i) *every triangle formed by* 3 *distinct z_i has area at least $\frac{1}{2}$,*
(ii) *we have*

$$\sum_{i<j} \frac{1}{|z_i - z_j|} \leqslant 8\sqrt{2}(p-1). \tag{4.2}$$

Proof. Statement (i) is an observation of Erdös [6, Appendix]; simply note that the area is half the value of a determinant that is not congruent to 0 (mod p).

For (ii), first observe the general inequality

$$(a^2+b^2)^{-\frac{1}{2}} \leqslant \sqrt{2}/(|a|+|b|). \tag{4.3}$$

Write $z_i = (x_i, y_i)$ and let $N(k)$ be the number of solutions of

$$|x_i - x_j| + |y_i - y_j| = k, \quad i<j. \tag{4.4}$$

Clearly the sum on the left side of (4.2) is bounded by

$$\sqrt{2} \sum_{k=1}^{2p-2} \frac{N(k)}{k}.$$

Now if

$$0 \leqslant a, b \leqslant p-1, \tag{4.6}$$

then the simultaneous equations

$$x_i - x_j \equiv a \pmod{p}, \qquad x_i^2 - x_j^2 \equiv b \pmod{p}, \tag{4.7}$$

have at most one solution $i<j$ modulo p. Hence $N(k) \leqslant 4k$ and the lemma follows. □

To prove our claimed result, it suffices to show that there is a way of placing $3n$ points inside the unit square so that the area covered by any vertex-disjoint triangulation is $\gg n^{-\frac{1}{2}}$.

In what follows, P is a set of $\approx n$ points with a distinguished subset Q of $\approx \sqrt{n}$ points. The cardinality of P shall be divisible by 3. It clearly suffices to construct such a set P in an $s \times s$ square Φ, where $s \approx \sqrt{n}$, so that every triangle of P with some vertex in Q has areas at least $\frac{1}{2}$ (the total area of these traingles is $\gg \sqrt{n}$, while the square has area $\ll n$).

Let p be an odd prime such that $p < \sqrt{n} \leqslant 2p$.

The square Φ shall be $[0, 100p] \times [0, 100p]$, and the distinguished subset Q shall be the set $\{z_1, \ldots, z_p\}$ of the lemma. The set P shall consist of Q together with p^2-p (or p^2-p+1 or p^2-p+2) points on a certain vertical line segment t such that no two are closer than $1/p$. For t we choose the rightmost vertical edge of Φ, i.e.,

$$t = \{(x, y) \colon x = 100p, 0 \leqslant y \leqslant 100p\}. \tag{4.8}$$

If all 3 vertices of a triangle δ lie in Q, then $A(\delta) \geqslant \frac{1}{2}$ by the lemma. If one vertex of δ lies in Q, it has area

$$\geqslant \tfrac{1}{2}(100-1)p(1/p) \geqslant \tfrac{1}{2}. \tag{4.9}$$

Finally, if 2 vertices of δ, say z_1, z_2, lie in Q, consider the line l joining them. If its slope exceeds 2 (say) in absolute value, the area δ of triangle $z_1z_2z_3$ for any z_3 on t is clearly very large. If the slope is less than 2 and q is the intersection of t and l, then

$$A(\delta) = A(\delta(z_1z_2z_3)) \geqslant \frac{|z_1 - z_2|\, h}{2(1^2+2^2)^{\frac{1}{2}}}, \tag{4.10}$$

provided every point of P on t is at least h units of distance above or below q. To ensure that each such $A(\delta)$ is at least $\frac{1}{2}$, it suffices to exclude from t a collection of subintervals of total length no more than

$$\sigma = 2 \sum_{i<j} \frac{\sqrt{5}}{|z_i - z_j|} \leqslant 16\sqrt{10}p < 64p, \tag{4.11}$$

by the lemma. Clearly enough remains of t to carry out the construction $(36p/(p^2-p+2) > 1/p)$, so the result follows.

5. Existence of a limit, and epsilon simplicity

For any integer q we have

$$3n = 3qm + 3r, \quad 0 \leq r < q. \tag{5.1}$$

Let

$$f(n) = a(n, P, \Sigma)/A(\Sigma). \tag{5.2}$$

The argument at the end of Section 2 shows that

$$f(n) \leq f(q)(1-\alpha) + \alpha \leq f(q) + \frac{1-f(q)}{m+1} \leq f(q) + \frac{1}{m+1}, \tag{5.3}$$

where, by (5.1), we have

$$m = [n/q]. \tag{5.4}$$

Hence

$$\limsup_{n \to \infty} f(n) \leq f(q), \tag{5.5}$$

and it follows immediately from (5.2) that $a(n, \Sigma)$ has a limit as $n \to \infty$.

We can make another use of (5.3). A statement that can be put in the form

$$x = 0 \tag{5.6}$$

shall be called *epsilon-simple* if knowledge of its truth (provided, say, by an oracle) enables us to explicitly write down a proof of

$$|x| < \epsilon \tag{5.7}$$

for any given rational $\epsilon > 0$ (the proof may be different for different values of ϵ). For example, a well-known though unpublished paper of J.B. Rosser establishes the epsilon-simplicity of the prime number theorem by means of the old Chebyshev method (see [2, pp. 578–581]).

Say Σ is the unit square. We show that the statement

$$\lim_{n \to \infty} a(n, \Sigma) = 0, \tag{5.8}$$

if true, is epsilon simple. By the continuity of the area of a triangle as a function of its vertices, we can compute $a(q, \Sigma)$, for any fixed constant value of q, to within any preassigned tolerance η. (Simply examine all sets of q triangle whose vertices lie on a rational grid with mesh size very small compared to η.) Since $m \to \infty$, as $n \to \infty$, the result follows from (5.3). Of course, given an ϵ, this crude method does not give us any *a priori* bound on the length of the proof as a function of ϵ.

6. Remarks

What makes this problem seemingly harder than Heilbronn's is the requirement of disjointness. However, if $3n$ points are in Euclidean 3 space (say n are red, n

are white and n are blue) and no 4 are coplanar, then there is a disjoint triangulation into tricolored triangles. This was shown nicely by means of the "Ham Sandwich Theorem" independently by Heuer, Goldstein and Winter [3].

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